Intersecting Curves in Projective Space: Bezout's Theorem

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1 Introduction

In this paper, we will look at the behavior of curves on the projective plane. Although we will focus on results in algebraically closed fields the following results only require that the intersection points exist in the projective plane.

1.1 The Projective Plane and Curves

We will start by giving algebraic definitions of affine and projective space. Throughout this paper, we will fix a field k that is algebraically closed.

Definition 1.1. For a field k the n dimensional projective space is

$$\mathbb{P}^n = \{[x_1 : x_2 : \cdots : x_{n+1}] | x_1, x_2, \ldots, x_{n+1} \text{ not all zero}\}/\sim$$

Which is isomorphic to the space $k^{n+1} - 0$ modded out by the relation that relates $[x_1 : x_2 : \cdots : x_{n+1}] \sim [y_1 : y_2 : \cdots : y_{n+1}]$ if and only if $x_i = ty_i$ for $t \in k - 0$ for all i.

Proposition 1.2. *The projective space $\mathbb{P}^n$ can be turned into the affine space $\mathbb{A}^n$ with the removal of any hyper plane in $\mathbb{P}^n$*

Example 1.3. When $n = 2$ we call $\mathbb{P}^2 = \{[x : y : z] | x, y, z \text{ not all zero}\}/\sim$ the projective plane. Notice that in this case we may remove the line $z = 0$ which means $z \neq 0$ and so up to scaling we may fix $z = 1$, which are the points of the form $[x : y : 1]$ which can be interpreted as the affine space k^2, with the bijection mapping $(x, y) \mapsto [x : y : 1]$.

Notice that because projective transformations take lines to lines we may assume any affine space can be interpreted as $\mathbb{A}^2 = \{[x : y : z] \in \mathbb{P}^2 | z = 1\} \cong k^2$ up to linear transformation.

Therefore the line that is removed to create $\mathbb{A}^2$, which we may assume to be $z = 0$ can be interpreted as the line at infinity, and behaves as a closure of $\mathbb{A}^2 \cong k^2$.

For example consider values $a, b \in k$ and consider the limit of a sequence of points in k^2 that is $\lim_{t\to\infty}(ta, tb)$. In projective space we may interpret this limit as $\lim_{t\to\infty}[ta : tb : 1]$ and with rescaling by t we may notice that $\lim_{t\to\infty}[ta : tb : 1] = \lim_{t\to\infty}[a : b : 1/t] = [a : b : 0]$ which is a point at infinity, encoding the direction in which the sequence approached infinity.

Now we will create definitions for curves in affine space which we can extend to curves in projective space.

Definition 1.4. A curve C in the affine plane $\mathbb{A}^2$ is the set of solutions to a polynomial equation $f(x, y) = 0$ where $f \in k[x, y]$. We may denote such a curve as $C : f(x, y) = 0$.

As in the definition of projective space, projective curves must be invariant under the scaling of the entries.

Definition 1.5. A projective curve C of degree d in $\mathbb{P}^2$ is the set of solutions to a non-constant polynomial equation $F(x, y, z) = 0$ where F is a *homogeneous polynomial* of degree d, that is $F \in k[x, y, z]$ is the sum of degree d monomials such that $F(tx, ty, tz) = t^d F(x, y, z)$.

Notice first that for every projective curve $C : F(x, y, z)$ we can extract an affine curve by removing a hyperplane, for example removing $z = 0$ and scaling $z = 1$. That is the affine part of F is the curve $f(x, y) = F(x, y, 1)$ where any projective point in affine space, $[a : b : 1]$ we have that $F(a, b, 1) = 0$ if and only if $f(a, b) = 0$. This process is called dehomogenization. Not all projective curves have affine components, for example, the curve $z = 0$ is only defined at infinity and has no non-constant dehomogenized polynomial in $k[x, y]$. However with the removal of another line ($x = 0$ for example), and therefore another perspective of $\mathbb{A}^2$, this curve would have an affine part.

Points at infinity, when $z = 0$, correspond to the slopes of the curve or how the curve approaches infinity, this can be formalized in a similar way as in the previous limit example.

Similar to the process of dehomogenization which extracts affine components of projective curves we can create a process of homogenization that takes an affine curve of degree d to a projective curve of degree d with a correspondence of its affine points. That is given an affine curve of degree d defined by the polynomial $f(x, y) = \sum_{i,j} a_{ij}x^i y^j$ we define the degree d homogenization as the polynomial $F(x, y, z) = \sum_{i,j} a_{ij}x^i y^j z^{d-i-j}$. Notice that $f(x, y) = F(x, y, 1)$, and so both curves share affine points. Furthermore, by construction $F(x, y, 0)$ is not identically zero for all x and y, meaning z is not a common factor and so F does not contain the line at infinity $z = 0$. The processes of homogenization and dehomogenization define a bijection between affine curves and projective curves that do not contain the line at infinity $z = 0$.

1.2 Intersecting Curves

In studying curves there is often the question of the number of intersecting points.

Definition 1.6. Given two curves $C_1 : F(x, y, z) = 0$ and $C_2 : G(x, y, z) = 0$ we say that a point $[a : b : c]$ is in the intersection $C_1 \cap C_2$ if $[a : b : c]$ is simultaneously a point of C_1 and C_2, that is $F(a, b, c) = G(a, b, c) = 0$.

A similar definition can be made for affine curves. In general, we are only interested in the case where the number of intersection points is finite, the case that two curves share no common components.

Definition 1.7. If $k[x, y]$ is a UFD then we may factor a projective curve as $F(x, y, z) = p_1(x, y, z)p_2(x, y, z) \cdots p_r(x, y, z)$ into the product of irreducible polynomial factors $p_i(x, y, y) \in k[x, y]$. We say that the *irreducible components* of C are the polynomials $p_i(x, y, z)$ for all i. We say two curves have no common components if their irreducible factors are distinct.

In our case where k is a field, $k[x, y]$ is always a UFD.

It is often the case that intersections will include multiplicity, for example, these are the cases where two curves share tangent lines. We may interpret the multiplicity to be the number of derivatives the curves share. Formally we define this notion with local rings.

Definition 1.8. Let k be algebraically closed, then let $K \leq k(x, y, z)$ with elements that are rational functions of the form $\Phi = F/G$, with F, G being homogeneous polynomials of x, y and z of the same degree. For a point $P \in \mathbb{P}^2$, we say that Φ is defined at P if $G(P) \neq 0$. And the local ring of a point P is the space $\mathcal{O}_P = \{\Phi \in K | \Phi \text{ is defined at } P\}$.

Notice that for projective curves defined by polynomials F_1, F_2 an ideal of $\mathcal{O}_P$ is defined as

$$\langle F_1, F_2 \rangle_P = \{F/G \in \mathcal{O}_P | F = H_1F_1 + H_2F_2\} \tag{1}$$

Where $H_1, H_2 \in k[X, Y, Z]$ such that F is homogeneous of degree equal to that of G. We can also restrict these constructions to the affine plane.

Proposition 1.9. *If $P = (a, b) = [a : b : 1] \in A$ then we may instead define $K \leq k(x, y)$ with elements that are rational functions of the form $\varphi = f/g$, with f, g being polynomials of x and y. For a point $P \in \mathbb{A}^2$, we say that φ is defined at P if $g(P) \neq 0$. And the local ring of a point P is the space $\mathcal{O}_P = \{\varphi \in K | \varphi \text{ is defined at } P\}$.*

Proof. These definitions coincide with definition 1.8 through the process of dehomoginization. □

Likewise for affine curves defined by polynomials f_1, f_2 an ideal of $\mathcal{O}_P$ is defined as $\langle f_1, f_2 \rangle_P = \mathcal{O}_P f_1 + \mathcal{O}_P f_2$ which agree with equation 1.

This then allows us to define the multiplicity of the intersection of two curves

Definition 1.10. For curves $C_1 : F_1(x, y, z) = 0$ and $C_2 : F_2(x, y, z) = 0$ and a point $P \in \mathbb{P}^2$ we have that

$$I(C_1 \cap C_2, P) = \dim(\mathcal{O}_P / \langle F_1, F_2 \rangle_P)$$

And if $P \in \mathbb{A}^2$ we have the alternative definition $I(C_1 \cap C_2, P) = \dim(\mathcal{O}_P / \langle f_1, f_2 \rangle_P)$ where f_1, f_2 represent the affine parts of F_1 and F_2.

Below we have given a few examples using this definition

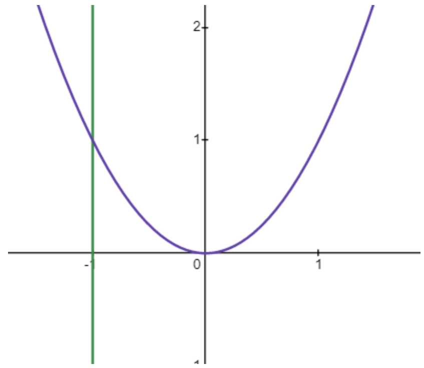

Figure 1: The intersection of $C_1 : x + 1 = 0$ shown in green and $C_2 : x^2 - y = 0$ in purple from example 1.11.

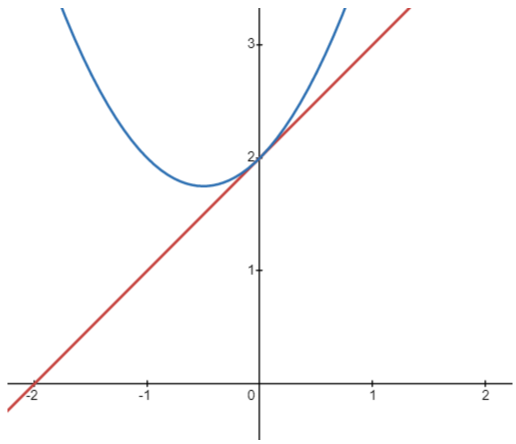

Figure 2: The intersection of $C_1 : x - y + 2 = 0$ shown in red and $C_2 : x^2 + x - y + 2 = 0$ in blue from example 1.12.

Example 1.11. Consider the projective curves $C_1 : F_1(x, y, z) = x + z = 0$ and $C_2 : F_2(x, y, z) = x^2 - yz = 0$, where C_1 is a line and C_2 is a quadratic. The restriction of both curves in the affine plane (from the removal of $z = 0$ and then scaling $z = 1$), where $f_1(x, y) = x + 1$ and $f_2(x, y) = x^2 - y$ is shown in figure 1. C_1 and C_2 intersect at the affine point $[-1 : 1 : 1]$ and the projective point $[0 : 1 : 0]$ encoding that the curves go off to infinity vertically. We will show that for the affine point $P = (-1, 1)$, that $I(C_1 \cap C_2, P) = 1$. To show this it suffices to show that $\langle f_1, f_2 \rangle_P = \mathcal{M}_P := \{\varphi \in \mathcal{O}_P | \varphi(P) = 0\}$, which can be done by showing $\langle f_1, f_2 \rangle_P$ contains all degree 1 polynomials that go through $(-1, 1)$. To see this notice that $(x - 1)(x + 1) = x^2 - 1$ and so $(x^2 - 1) - (x^2 - y) = y - 1$. This means that $x + 1, y - 1 \in \langle f_1, f_2 \rangle_P$ which generates all polynomials that go through $(-1, 1)$.

Example 1.12. Now we will show an example with an intersection of higher multiplicity to motivate an interpretation of multiplicity. Consider the projective curves $C_1 : F_1(x, y, z) = x - y + 2z = 0$ and $C_2 : F_2(x, y, z) = x^2 + xz - yz + 2z^2 = 0$, where C_1 is a line and C_2 is a quadratic. The restriction of both curves in the affine plane (from the removal of $z = 0$ and then scaling $z = 1$), where $f_1(x, y) = x - y + 2$ and $f_2(x, y) = x^2 + x - y + 2$ is shown in figure 2. C_1 and C_2 intersect at the affine point $[0 : 2 : 1]$. Notice that because $x^2 + x - y + 2 - (x - y + 2) = x^2$ we have that $\langle f_1, f_2 \rangle_P = \langle x - y + 2, x^2 \rangle_P$. Notice that this means $\mathcal{O}_P / \langle f_1, f_2 \rangle_P$ is spanned by the parallel classes of 1 and x, and so has dimension 2. To see this another way we may consider the Taylor expansions of both curves with respect to y, that is $y = 2 + 1 \cdot x$ and $y = 2 + 1 \cdot x + 1 \cdot x^2$, where we see that both curves share a derivatives at the point of intersection. This motivates that the intersection multiplicity counts the number of shared derivatives. However this interpretation does not extend to all fields, such as finite fields.

Example 1.13. Now we will show an example with an intersection of multiplicity 3. Consider the projective curves $C_1 : F_1(x, y, z) = z^3 + x^3 - yz^2 = 0$ and $C_2 : F_2(x, y, z) = z - y = 0$. The restriction of both curves in the affine plane (from the removal of $z = 0$ and then scaling $z = 1$), where $f_1(x, y) = 1 + x^3 - y$ and $f_2(x, y) = 1 - y$ is shown in figure 3. C_1 and C_2 intersect at the affine point $[0 : 1 : 1]$. Notice that because $1 + x^3 - y - (1 - y) = x^3$ we have that $\langle f_1, f_2 \rangle_P = \langle 1 - y, x^3 \rangle_P$. Notice that this means $\mathcal{O}_P / \langle f_1, f_2 \rangle_P$ is spanned by the parallel classes of 1, x and x^2, and so has dimension 3. We may also see that both curves share a derivatives and a second derivative at the point of intersection. This again shows that the intersection multiplicity counts the number of shared derivatives.

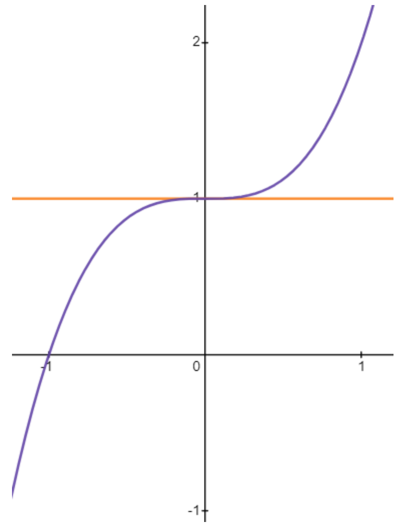

Figure 3: The intersection of $C_1 : 1 + x^3 - y = 0$ shown in purple and $C_2 : 1 - y = 0$ in orange from example 1.13.

The following are expected facts about the intersection multiplicity

Proposition 1.14. *$I(C_1 \cap C_2, P)$ is a non-negative finite integer and $P \in C_1 \cap C_2$ if and only if $I(C_1 \cap C_2, P) \geq 1$*

Proof. A proof of the first part of this statement is given in propositon 2.7 □

Now we will explore the intersection of curves with Bezout's Theorem

Theorem 1.15. *(Bezout's theorem) Let C_1 and C_2 be projective curves of degree n_1 and n_2 respectively with no common components then*

$$\sum_{P \in C_1 \cap C_2} I(C_1 \cap C_2, P) = n_1 n_2$$

If every point in the intersection has an intersection of multiplicity 1 then it is the case that $\#(C_1 \cap C_2) = n_1 n_2$. In all cases $\#(C_1 \cap C_2) \leq n_1 n_2$

We will prove Bezout's theorem throughout the next section as outlined in [2].

2 A proof of Bezout's Theorem

To prove Bezout's theorem we first prove a series of lemmas restricting the curves to the affine plane, the points on the curves in $\mathbb{A}^2$ and then extending the theorem to the projective case using the idea from proposition 1.2.

2.1 The Affine Case

Throughout the following lemmas, we will assume that $C_1 : f_1(x, y) = 0$ and $C_2 : f_2(x, y) = 0$ are affine curves in $\mathbb{A}^2$ for some algebraically closed field k, with no common components, with $n_1 = \deg(f_1)$ and $n_2 = \deg(f_2)$. We will also define the polynomial ring in two variables to be $R := k[x, y]$ of which both f_1 and f_2 live and we will consider the ideal $\langle f_1, f_2 \rangle = Rf_1 + Rf_2$. We will first look at the space $R/\langle f_1, f_2 \rangle$, which is a k-vector space.

Lemma 2.1. *Restricting the curves C_1 and C_2, to $\mathbb{A}^2$ we find that*

$$\#(C_1 \cap C_2 \cap \mathbb{A}^2) \leq \dim (R/\langle f_1, f_2 \rangle)$$

Proof. Let $P_1, P_2, \ldots, P_m$ be the distinct points in the intersection $C_1 \cap C_2$. Fix a point P_i and for any other point $P_j \neq P_i$ we may construct a line $\ell_{i,j}(x, y)$ such that $\ell_{i,j}(P_i) \neq 0$ and $\ell_{i,j}(P_j) = 0$ (such a construction could be the line going through P_j that is perpendicular to the line determined by P_i and P_j). This allows us to construct a polynomial $h_i(x, y) = \frac{\prod_{j \neq i} \ell_{i,j}(x,y)}{\prod_{j \neq i} \ell_{i,j}(P_i)}$, which satisfies that $h_i(P_i) = 1$ and $h_i(P_j) = 0$ for all $j \neq i$.

To prove the lemma we will show that the polynomials h_i for all i are linearly independent in $R/\langle f_1, f_2\rangle$. To see that assume that

$$c_1h_1 + c_2h_2 + \cdots + c_mh_m = g_1f_1 + g_2f_2, \text{ for } g_1, g_2 \in R \tag{2}$$

Notice however that for the point P_i we have that equation 2 gives us $c_i = 0$ as $h_j(P_i) = 0$ for $i \neq j$ by the construction of h_j. And so there are at least as linearly independent polynomials in $R/\langle f_1, f_2\rangle$ as there are points of intersection. □

Lemma 2.2. *Let R_d be the k-vector space of polynomials in R of degree $\leq d$. The* $\dim(R_d) = \frac{1}{2}(d+1)(d+2)$. *Furthermore for any non-zero polynomial $f \in R$,* $\dim(R_df) = \dim(R_d)$

Proof. R_d is the vector space spanned by all monomials of degree at most d. For any fixed degree $k < 1$ there are $\binom{k+1}{1} = k+1$ such monomials, which follow from a multi-choose combinatorial argument. This means

$$\dim(R_d) = \sum_{k=0}^{d} k + 1 = \sum_{k=1}^{d+1} k = \frac{1}{2}(d+1)(d+2)$$

Now to prove the second claim we will construct an isomorphism between $R_d \to R_df$ such that $g \mapsto fg$. To show this we need only show that fx^iy^j for all $i + j \leq d$ forms a basis. This is a spanning set as the monomials span R_d, and is linearly independent as the monomials are linearly independent, and in any linear combination, we may factor out f. □

Now we will look at the k-vector space $W_d = \{g_1f_1 + g_2f_2 | g_1 \in R_{d-n_1}, g_1 \in R_{d-n_2}\}$. Notice that W_d contains polynomials of degree $\leq n$ and so is a subspace of R_d.

Lemma 2.3.

$$\dim(R/\langle f_1, f_2\rangle) \leq n_1n_2$$

Proof. Consider a collection of $n_1n_2 + 1$ polynomials $g_1, g_2, \ldots, g_{n_1n_2+1} \in R$ which we will show are linearly dependent in $R/\langle f_1, f_2\rangle$. First, let d be the maximum of $n_1 + n_2$ and the degree of the $n_1n_2 + 1$ polynomials, and consider the subspace R_d, which contains all the polynomials. We will show that the polynomials are linearly dependent in R_d/W_d by dimensional arguments.

First notice that $W_d = R_{d-n_1}f_1 \cup R_{d-n_2}f_2$ and because $d \geq n_1 + n_2$ we have that any element $h \in R_{d-n_1}f_1 \cap R_{d-n_2}f_2$ can be decomposed as g_1f_1 and g_2f_2, and where it must be the case that $f_1|g_2$ and $f_2|g_1$, as f_1 and f_2 share no common factors. And so f_1f_2 is a common factor of h, meaning $h = g_3f_1f_2$ where $g_3 \in R_{d-n_1-n_2}$. This shows that $R_{d-n_1}f_1 \cap R_{d-n_2}f_2 = R_{d-n_1-n_2}f_1f_2$. Notice that this allows us to determine that

$$\begin{aligned}\dim(W_d) &= \dim(R_{d-n_1}f_1) + \dim(R_{d-n_2}f_2) - \dim(R_{d-n_1}f_1 \cap R_{d-n_2}f_2)\\ &= \dim(R_{d-n_1}f_1) + \dim(R_{d-n_2}f_2) - \dim(R_{d-n_1-n_2}f_1f_2)\end{aligned}$$

and from lemma 2.2 we have that

$$\begin{aligned}\dim(R_d/W_d) &= \dim(R_d) - \dim(W_d)\\ &= \dim(R_d) - \dim(R_{d-n_1}f_1) - \dim(R_{d-n_2}f_2) + \dim(R_{d-n_1}f_1 \cap R_{d-n_2}f_2)\\ &= \frac{1}{2}((d+1)(d+2) - (d-n_1+1)(d-n_1+2) - (d-n_2+1)(d-n_2+2) + (d-n_1-n_2+1)(d-n_1-n_2+2))\\ &= n_1n_2\end{aligned}$$

Meaning for the polynomials $g_1, g_2, \ldots, g_{n_1n_2+1} \in R_d \subseteq R$ there must exists a non-trivial linear combination such that $\sum_{j=1}^{n_1n_2+1} c_jg_j \in W_d$, as the dimension of R_d/W_d is n_1n_2. This also proves that because $W_d \subseteq \langle f_1, f_2\rangle$ that $\sum_{j=1}^{n_1n_2+1} c_jg_j \in \langle f_1, f_2\rangle$ as well, meaning we there exists a non-trivial linear independence equation for $R/\langle f_1, f_2\rangle$ as well. And so $\dim(R/\langle f_1, f_2\rangle) \leq n_1n_2$. □

In the previous proofs, we have looked at curves restricted to $\mathbb{A}^2$, the affine plane and we have shown a weak inequality that $\#(C_1 \cap C_2 \cap \mathbb{A}^2) \leq \dim(R/\langle f_1, f_2\rangle) \leq n_1n_2$. Now we will work to strengthen this equality and then extend it to all of the projective plane. First given a curve $C : f(x, y) = \sum_{i,j} c_{ij}x^iy^j = 0$ where we want to analyze the points at infinity of C. Assume that the degree of f is n and notice that when we homogenize f we get

$$f(x, y, z) = \sum_{i+j=n} c_{ij}x^iy^j + O(z)$$

where $O(z)$ denotes all terms with a z. In this case, to study points at infinity we will look at $z = 0$ where we find that $f^*(x, y) := f(x, y, 0) = \sum_{i+j=n} c_{ij} x^i y^j$. Because k is algebraically closed we may factor f^* are a product of linear terms giving $f^*(x, y) = \prod_{i=1}^{n}(a_i x + b_i y)$ for $a_i, b_i \in k$ and a_i and b_i not both zero. This means the roots of f^* and therefore the points at infinity of f are when $a_i x = -b_i y$. Notice that the point $(b_i, -a_i)$ and all multiples satisfy this and so the point at infinity are of the form $[b_i : -a_i : 0]$ for all i.

Lemma 2.4. *Assume that the curves C_1 and C_2 do not meet at infinity. Then f_1^* and f_2^* share no common factors, where f^* denotes the sum of the terms of highest degree in f.*

Proof. From the previous observation we noted that we could decompose $f_1^*(x) = \prod_{i=1}^{n}(a_i x + b_i y)$ for $a_i, b_i \in k$ and a_i and b_i not both zero. And likewise we could decompose $f_2^*(x) = \prod_{i=1}^{n}(c_i x + d_i y)$ for $c_i, d_i \in k$ and c_i and d_i not both zero.

Now we will assume that f_1^* and f_2^* do have a common factor meaning there exists some ℓ and k such that $a_\ell = c_k$ and $b_\ell = d_k$. And because the points at infinity of C_1 are $[b_i : -a_i : 0]$ for all i and the points at infinity of C_2 are $[d_j : -c_j : 0]$ for all j we know that both curves intersect at $[b_\ell : -a_\ell : 0] = [d_k : -c_k : 0]$. The statement follows from the contrapositve. $\square$

Lemma 2.5. *Assume that for the curves C_1 and C_2 that f_1^* and f_2^* share no common factors Then $\langle f_1, f_2\rangle \cap R_d = W_d$ for all $d \geq n_1 + n_2$.*

This lemma is proven in a rather interesting way, looking at the intersections at infinity to argue about the degree of the functions at infinity.

Proof. First one direction is trivial, as all vectors in W_d are of degree $\leq d$ and furthermore we know that $W_d \subseteq \langle f_1, f_2\rangle$ so we have that $W_d \subseteq \langle f_1, f_2\rangle \cap R_d$.

For the other direction consider an element $f \in \langle f_1, f_2\rangle \cap R_d$, such that $f = g_1 f_1 + g_2 f_2$ for g_1, g_2 of smallest possible degree. Notice that g_1, g_2 may be functions of an arbitrarily big degree, which would lead to the cancellation in the sum. We will show that this need not be the case, so assume that $\deg(g_1) > d - n_1$ and looking at the terms of highest degree it must be the case that $(g_1 f_1)^* + (g_2 f_2)^* = g_1^* f_1^* + g_2^* f_2^* = 0$, as otherwise the degree of f would be $> d$. Likewise we also know that $\deg(g_1 f_1) = \deg(g_2 f_1) = m > d \geq n_1 + n_2$. Because f_1^* and f_2^* have no common factors is must be the case that $f_1^* | g_2^*$ and $f_2^* | g_1^*$ as both f_1^* and f_2^* both divide $g_1^* f_1^* = -g_2^* f_2^*$. This means there exists some function h_1 and h_2 such that $g_1^* = h_1 f_2^*$ and $g_2^* = h_2 f_1^*$. Furthermore notice that this gives us that $h_1 f_2^* f_1^* = -h_2 f_1^* f_2^*$, and because R as a ring has no zero divisors we know that $h_1 = -h_2$.

Using this construction we know that $g_1^* f_1^* + g_2^* f_2^* = h_1 f_2^* f_1^* + g_2^* f_2^* = 0$ and so $h_1 f_1^* + g_2^* = 0$, meaning the polynomial $h_1 f_1 + g_2$ has no $\deg(g_1^*)$ terms and therefore $\deg(h_1 f_1 + g_2) < \deg(g_2)$. Along the same arguments we can observe that $g_1^* f_1^* + g_2^* f_2^* = g_1^* f_1^* + h_2 f_1^* f_2^* = 0$ and so $g_1^* + h_2 f_2^* = 0$, meaning the polynomial $g_1 + h_2 f_2$ has no $\deg(g_1^*)$ terms and therefore $\deg(g_1 + h_2 f_2) < \deg(g_1)$. Using this construction we can also observe that

$$(g_1 + h_2 f_2) f_1 + (h_1 f_1 + g_2) f_2 = g_1 f_1 + h_2 f_2 f_1 + h_1 f_1 f_2 + g_2 f_2 = f + f_1 f_2 (h_1 + h_2) = f$$

This contradicts that g_1 and g_2 were chosen minimally. And so there must exists choices of g_1 and g_2 with $\deg(g_1) \leq d - n_1$ and $\deg(g_2) \leq d - n_2$. So therefore $f \in W_d$ $\square$

Lemma 2.6. *Assume that the curves C_1 and C_2 do not meet at infinity. Then* $\dim(R/\langle f_1, f_2\rangle) = n_1 n_2$

Proof. First, because the curves C_1 and C_2 do not meet at infinity and from the lemmas 2.4 and 2.5 we know that it must be the case that $\langle f_1, f_2\rangle \cap R_d = W_d$ for all $d \geq n_1 + n_2$. We also know that from lemma 2.1 that $\dim(R/\langle f_1, f_2\rangle) \leq n_1 n_2$ so it suffices to show that $\dim(R/\langle f_1, f_2\rangle) \geq n_1 n_2$. Notice that in lemma 2.3 we know that $\dim(R_d/W_d) = n_1 n_2$. So consider a collection of linearly independent vectors $g_1, g_2, \ldots, g_{n_1 n_2}$. And notice that any linear combination of these vectors would have a degree at most d, and because there is no linear combination over k that results in a vector in $W_d = \langle f_1, f_2\rangle \cap R_d$ these vectors must also be linearly independent in $R/\langle f_1, f_2\rangle$. This means that $\dim(R/\langle f_1, f_2\rangle) \geq n_1 n_2$, proving the lemma. $\square$

Proposition 2.7. *If C_1 and C_2 have no common components then for $P \in C_1 \cap C_2 \cap \mathbb{A}^2$ we have that $I(C_1 \cap C_2, P)$ is finite*

Proof. Fix a point $P \in \mathbb{A}^2$ and notice that for two functions $g/h, g'/h' \in \mathcal{O}_P$ we can rewrite them as gh'/hh' and $g'h/hh'$ sharing a common denominator. This can be repeated inductively such that any collection of rational functions share a common denominator. So consider a collection of rational function $g_1/h, g_2/h, \ldots, g_r/h$ and notice that any linear combination can be rewritten as $a_1g_1/h + a_2g_2/h + \cdots + a_rg_r/h = \frac{a_1g_1+a_2g_2+\cdots+a_rg_r}{h}$. And so if the polynomials $g_1, g_2, \ldots, g_r$ were linearly dependent in $R/\langle f_1, f_2\rangle$, meaning $a_1g_1 + a_2g_2 + \cdots + a_rg_r \in \langle f_1, f_2\rangle$ then it would also be the case that $\frac{a_1g_1+a_2g_2+\cdots+a_rg_r}{h} \in \langle f_1, f_2\rangle_P$, and so the rational function $g_1/h, g_2/h, \ldots, g_r/h$ would be linearly dependent. This means by contrapositive that any collection of rational function $g_1/h, g_2/h, \ldots, g_r/h$ that are linearly independent in $\mathcal{O}_P/\langle f_1, f_2\rangle_P$ would result in the polynomials $g_1, g_2, \ldots, g_r$ being linearly independent in $R/\langle f_1, f_2\rangle$. This shows that $I(C_1 \cap C_2, P) = \dim(\mathcal{O}_P/\langle f_1, f_2\rangle_P) \leq \dim(R/\langle f_1, f_2\rangle) \leq n_1n_2$. □

Now we want the inequalities we have built so far to include the multiplicity of the intersections, defined in definition 1.10

Lemma 2.8. *If C_1 and C_2 have no common components then*

$$\sum_{P \in C_1 \cap C_2 \cap \mathbb{A}^2} I(C_1 \cap C_2, P) = \dim(R/\langle f_1, f_2\rangle)$$

Proof. We will provide a proof sketch. First for the $\leq$ direction. There exists a well defined epimorphism

$$R \to \prod_{P \in C_1 \cap C_2 \cap \mathbb{A}^2} \mathcal{O}_P/\langle f_1, f_2\rangle_P$$
$$f \mapsto (\ldots, f \mod (f_1, f_2)_P, \ldots)_{P \in C_1 \cap C_2 \cap \mathbb{A}^2}$$

Let J be the kernel of the space and therefore

$$\dim(R/J) = \sum_{P \in C_1 \cap C_2 \cap \mathbb{A}^2} \dim(\mathcal{O}_P/\langle f_1, f_2\rangle_P) = \sum_{P \in C_1 \cap C_2 \cap \mathbb{A}^2} I(C_1 \cap C_2, P)$$

Notice also that $\langle f_1, f_2\rangle \subseteq J$ meaning $\dim(R/J) \leq \dim(R/\langle f_1, f_2\rangle)$ proving the first direction. Now we need only show that $J \subseteq \langle f_1, f_2\rangle$. Fix some $f \in J$ and consider the ideal $L = \{g \in R | gf \in \langle f_1, f_2\rangle\}$ and we will show that $1 \in L$ which implies $\langle f_1, f_2\rangle = J$. Notice first the L is an ideal which follows from the fact that $\langle f_1, f_2\rangle$ is an ideal, and so if $g_1, g_2 \in L$ then $(g_1 + g_2)f \in \langle f_1, f_2\rangle$ and for $h \in R$ we have that $hg_1f \in \langle f_1, f_2\rangle$. Likewise L has the property that for all $P \in \mathbb{A}^2$ there exists some $g \in L$ such that $g(P) \neq 0$, which can be shown, with the initial assumption that $1 \notin L$, leads to a contradiction that $1 \in L$. And so L being an ideal containing 1, must mean that $L = R$, and so this must imply that $J \subseteq \langle f_1, f_2\rangle$ proving the statement. □

Now putting the previous lemmas together we have shown a weaker version of Bezout's theorem

Proposition 2.9. *If C_1 and C_2 have no common components and do no intersect at infinity then*

$$\sum_{P \in C_1 \cap C_2} I(C_1 \cap C_2, P) = n_1n_2$$

2.2 Curves that Meet at Infinity

Recall from proposition 1.2 that an affine plane can be defined by the removal of any line in $\mathbb{P}^2$ and therefore any such choice of line, can be interpreted as the line $z = 0$ up to projective transformation. This means if C_1 and C_2 do in fact intersect at infinity, or on the line $z = 0$ we need only change perspectives to a new line at infinity, in which the curves C_1 and C_2 do not intersect.

Lemma 2.10. *For projective curves C_1 and C_2 with no common components there exists a line L such that L does not contain any of the points in $C_1 \cap C_2$.*

Proof. First we want to show that $C_1 \cap C_2$ is finite. Let L be a line that is not a component of C_1 or C_2, and consider the affine plane $\mathbb{A}^2$ constructed by its removal we know that $\sum_{P \in C_1 \cap C_2 \cap \mathbb{A}^2} I(C_1 \cap C_2, P) \leq n_1n_2$, meaning there are only a finite number of points of $C_1 \cap C_2$ not on L. Likewise because there are infinitely many lines, as k is its self infinite there must exists a second line L' that is distinct from L and shares no components with C_1 and C_2. And similarly for L' there must also be a finite number of points in $C_1 \cap C_2$ not on L'. And because L and L' intersect as a single point the total number of points in $C_1 \cap C_2$ must be finite.

Given that $C_1 \cap C_2$ is finite and knowing that k is algebraically closed and therefore not finite its self we will show that there is a line L not meeting any of the points of $C_1 \cap C_2$. There must exist at least one point on the line $z = 0$ that is not in $C_1 \cap C_2$ as $z = 0$ is not a common component, call it $[\beta : \alpha : 0]$ and let L be the line $\alpha x + \beta y + \gamma z = 0$ and consider the affine points of the form $[x : y : 1]$, and pick γ to be any value in k that is no equal to $-\alpha x - \beta y$ for all affine points $[x : y : 1]$ in $C_1 \cap C_2$. This is possible as k has infinitely many elements and $C_1 \cap C_2$ has finitely many. $\square$

Notice that such a line L must also not be a component of either C_1 and C_2, as otherwise there would be a intersection point on the line. And now together with section 2.1 we can prove Bezout's Theorem

Theorem 2.11. *(Bezout's Theorem) Let C_1 and C_2 be projective curves of degree n_1 and n_2 respectively with no common components, then*

$$\sum_{P \in C_1 \cap C_2} I(C_1 \cap C_2, P) = n_1 n_2$$

Proof. By lemma 2.10 we know that there must exist some line L that does not contain any of the points in $C_1 \cap C_2$. Use this L as the line at infinity and remove it to create the affine plane $\mathbb{A}^2$. Now using lemma 2.9 and the fact that L is not a component of either C_1 or C_2 we know that the corresponding affine curve has the same degree and so we get the result as desired. $\square$

3 Testing For Common Components

Often Bezout's theorem can be used to determine when two curves are the same or share common components, even in non-algebraically closed fields. Consider the two projective curves over the finite field $\mathbb{F}_3$ restricted to the affine plane where $z = 1$.

$$f(x, y) = x^2 + xy - x^2y - xy^2$$

$$g(x, y) = x^2 + 2x - x^2y - 2xy$$

We want to determine if these curves have a common component. Bezout's theorem (assuming z is not a common component of either curve) says that in $\overline{\mathbb{F}_3}$ the algebraic closure of $\mathbb{F}_3$ there should be 9 total points up to multiplicity.

Immediately we may rule out the possibility that these curves are the same. Notice that $f(1, 0) = 1$ and $g(1, 0) = 0$, however they may still contain a common component. We can test all point in $\mathbb{F}_3^2$ and find that the points of intersection in the affine plane of which there are 6 are $(0, 0), (0, 1), (0, 2), (1, 1), (1, 2)$, and $(2, 1)$.

Now consider the field extension $\mathbb{F}_9 = \mathbb{F}_3[x]/\langle x^2 + 1\rangle$ which effectively adds in the value x acting as $\sqrt{-1} = i$. This give us additional points of intersection: $(0, i), (i, 0), (0, 1 + i), (0, 2 + i)$, and more. However notice that this gives us 10 points of intersection meaning f and g must have a common component.

References

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